Quantum Proofs of Proximity

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Introduction

Part I: Algorithms

Part II: Complexity separations

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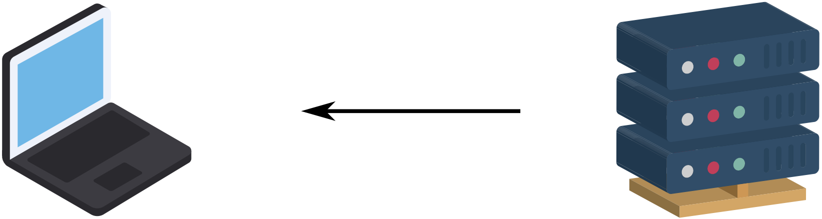
Massive datasets, IoT: devices with *dramatically* different power.



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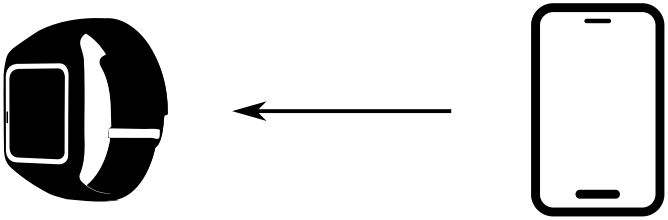

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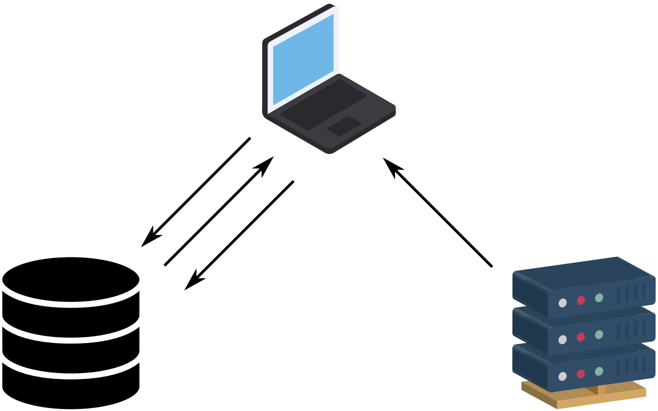
Delegation of computation: *prover* computes, *verifier* checks.
Efficient: $\tilde{O}(n)$ verifier runtime.

Proofs. . .



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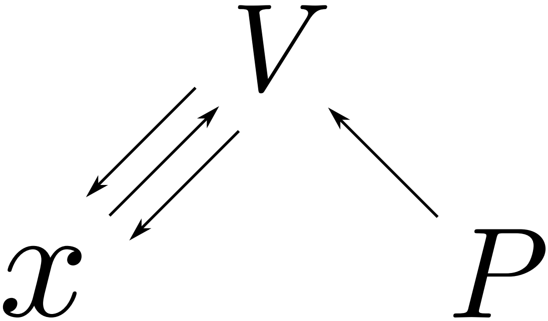

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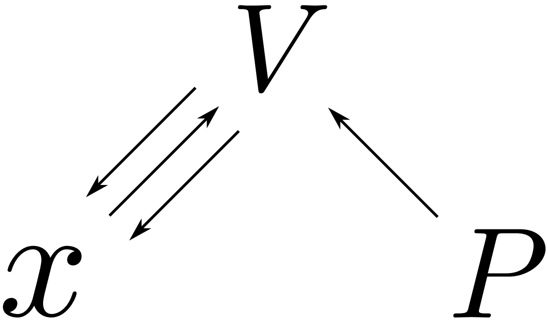

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Query complexity q, proof complexity p.

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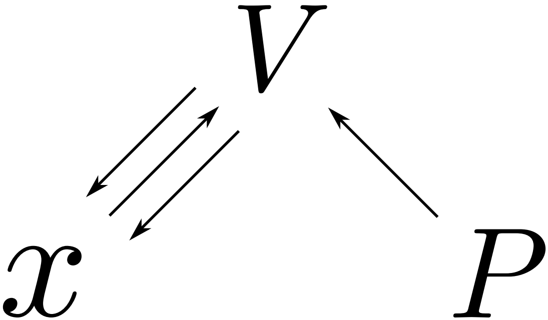

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Efficient: $o(n)$ queries and proof length.

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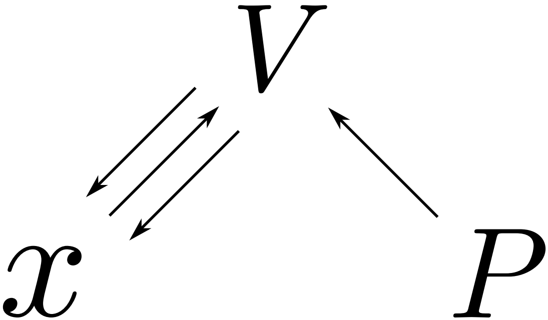
Example: Does $x \in \{0,1\}^n$ contain a 1?



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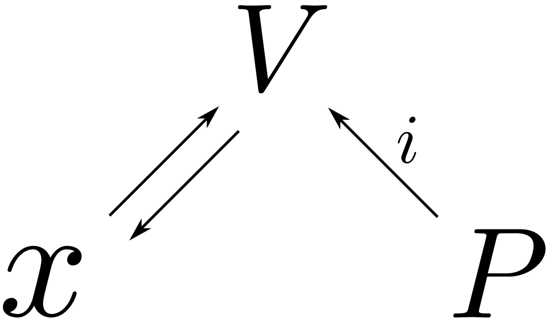

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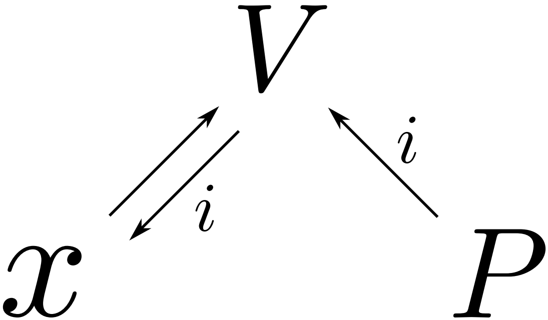

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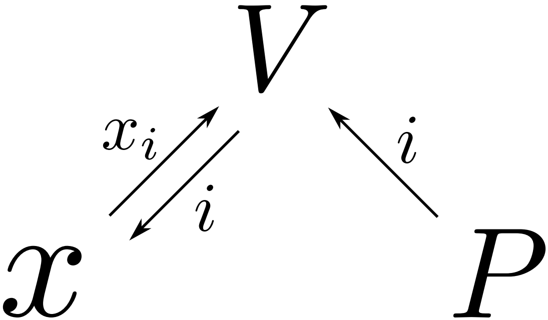

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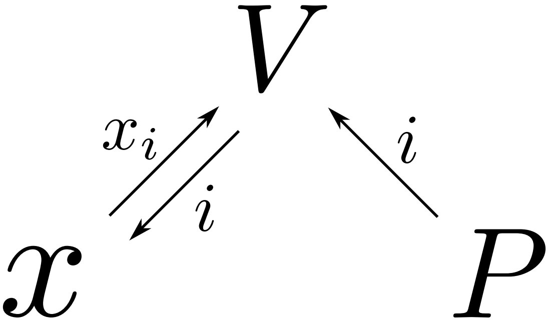
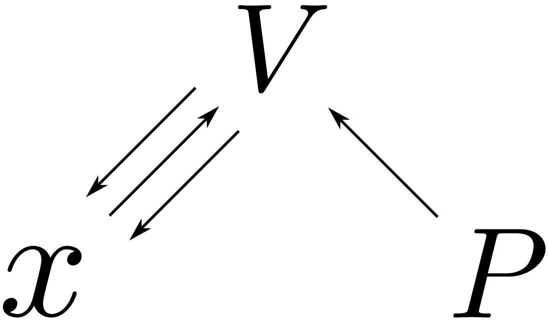


Query complexity 1, proof complexity $\log n$.
($\Omega(n)$ with no proof!)

. . . of proximity. . .



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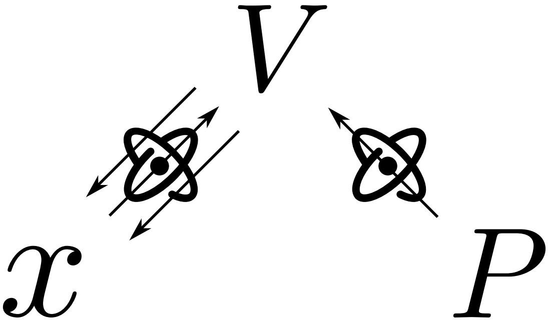
Query complexity q, proof complexity p.



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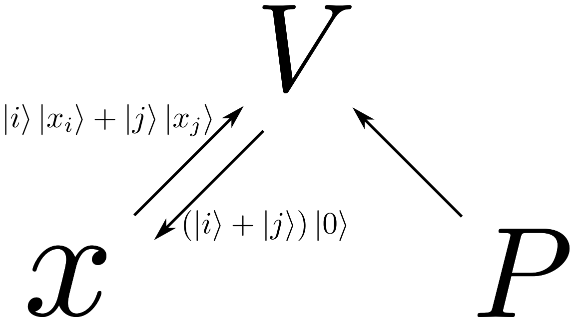
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$|i\rangle |x_i\rangle + |j\rangle |x_j\rangle$
$(|i\rangle + |j\rangle) |0\rangle$
x
P

Introduction

Part I: Algorithms

Part II: Complexity separations

Theorem (Amplitude amplification [Brassard et al., 2002])

If a one-sided randomised algorithm makes q queries and detects an error with probability ρ, there is a quantum algorithm making $O(q/\sqrt{\rho})$ queries that succeeds w. p. $2/3$.

Problem: Verify if $x \in \{0,1\}^n$ has even parity.

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Quantumly, an $n^{2/3}$-bit proof and $O(n^{2/3})$ queries suffice!

Problem: Verify if $x \in \{0,1\}^n$ has even parity.

$x =$ 

If the parity of the proof π is odd, reject.
Sample $i \in [p]$ uniformly and query the i^{th} block of n/p bits. Accept if their parity matches π_i, reject otherwise.

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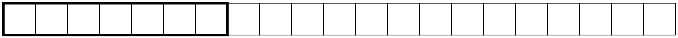
$x =$ 

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$$\Downarrow$$

Seting $p = n^{2/3}$ in the previous algorithm, it makes $n^{1/3}$ queries to detect an error w. p. $1/n^{2/3}$. Therefore,

$$q = O\left(\frac{n^{1/3}}{\sqrt{1/n^{2/3}}}\right) = O(n^{2/3}).$$

Main result

Theorem

A similar strategy works for every decomposable property.

(Includes k-monotonicity, acceptance by branching programs, membership in context-free languages, and Eulerian graph orientations.)

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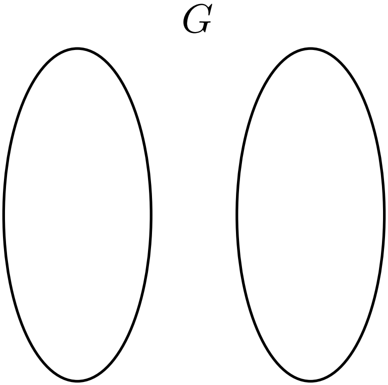

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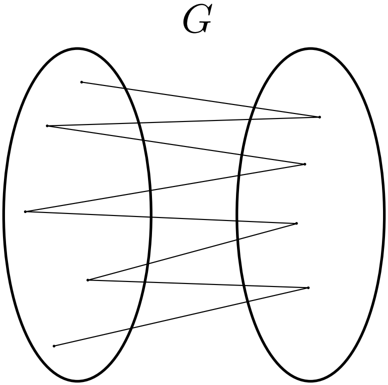

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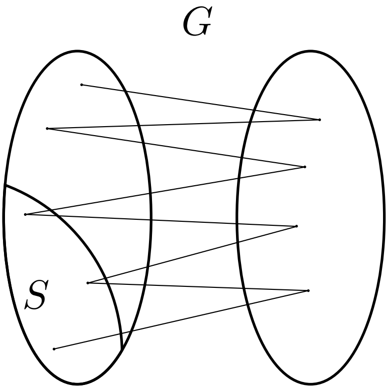

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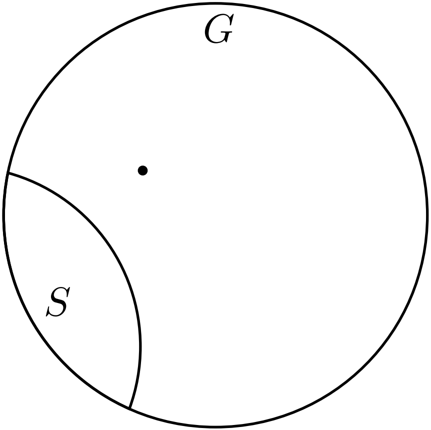

Sample a vertex, take many random walks from it and check whether any pair of walks ends at S with different parities. If so, reject, and accept otherwise.

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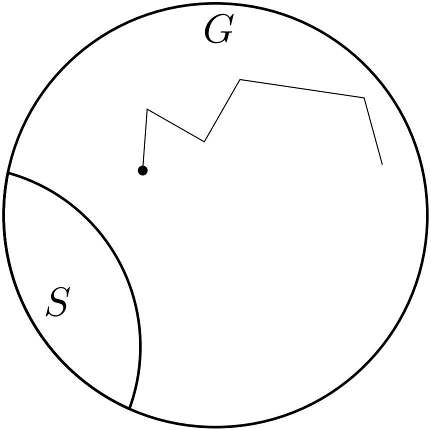

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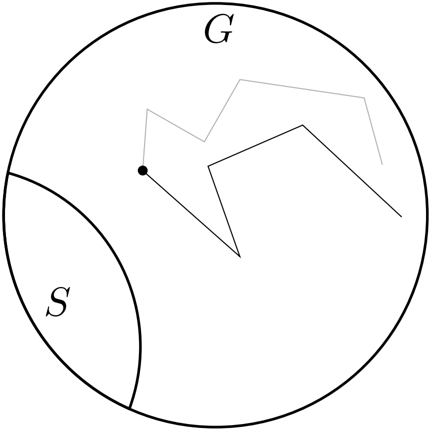

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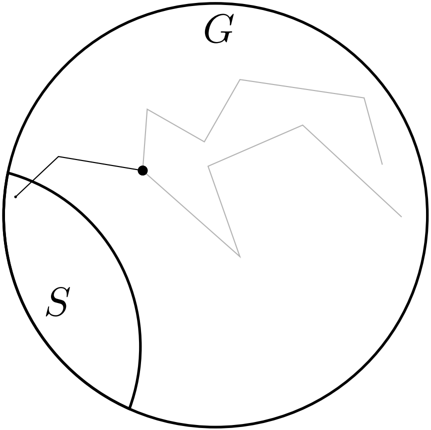

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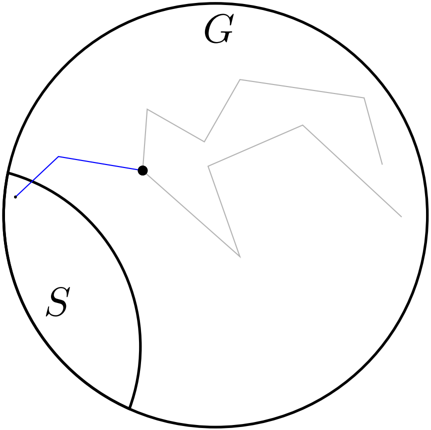

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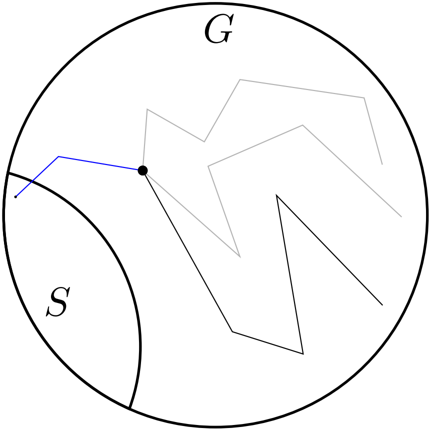

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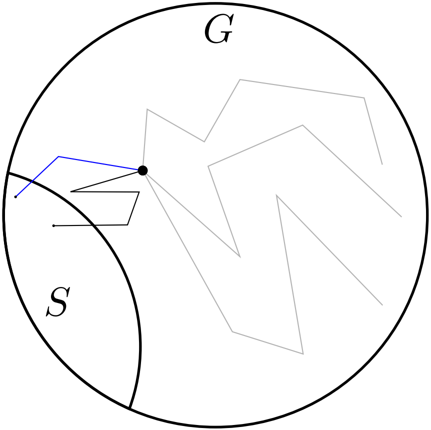

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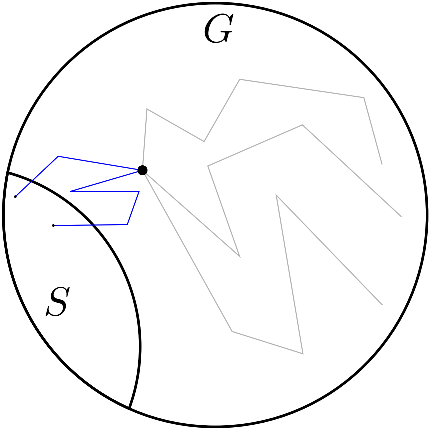

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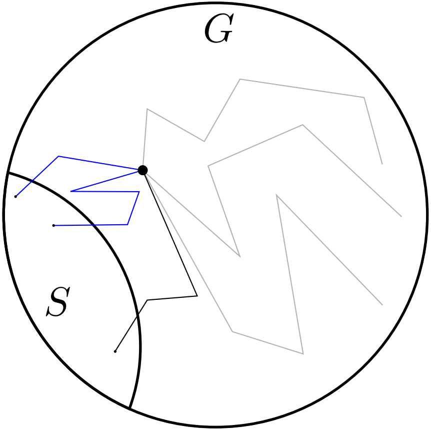

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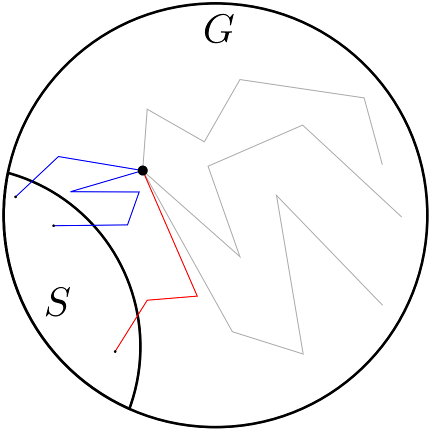

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Theorem

For any k, bipartiteness of rapidly-mixing n-vertex graphs (in the bounded-degree model) admits quantum proofs of proximity with $k\log(n)$-bit proofs and query complexity $\tilde{O}\left(\left(\frac{n}{k\varepsilon^2}\right)^{1/3}\right)$.

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Best quantum tester makes $O(n^{1/3})$ queries, so $k \approx \sqrt{n}$ beats it.
In best classical proof of proximity, $k \approx n^{2/3}$ for $O(n^{1/3})$ queries.
(Also improves dependence on ε.)

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Decomposable properties: known *classical* proofs of proximity
[Gur and Rothblum, 2018, Goldreich et al., 2018]

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Part II: Complexity separations

Complexity classes

	V	$V \leftarrow P$	$V \leftrightarrow P$
Classical	$\mathcal{PT}$	$\mathcal{MAP}$	$\mathcal{IPP}$
Quantum	$\mathcal{QPT}$	$\mathcal{QMAP}$	$\mathcal{QIPP}$

Also $\mathcal{QCMAP}$: classical proof, quantum input access.

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Also $\mathcal{QCMAP}$: classical proof, quantum input access.

$\mathcal{C} := \mathcal{C}(\varepsilon, p, q)$ with $p, q = \mathrm{polylog}(n)$ and

ε a small enough constant.

Main result

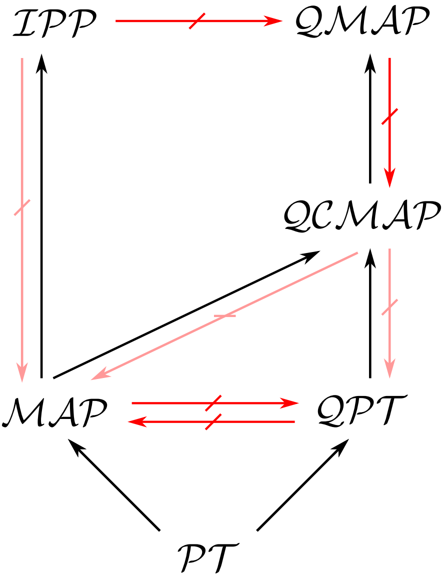
Theorem

The following separations hold:

- $\mathcal{QMAP} \not\subseteq \mathcal{MAP} \cup \mathcal{QPT}$, *i.e., quantum input access with a proof are more powerful in tandem than separately;*
- $\mathcal{QMAP} \not\subseteq \mathcal{QCMAP}$, *i.e., classical proofs are weaker than quantum even with a quantum verifier;*
- $\mathcal{IPP} \not\subseteq \mathcal{QMAP}$, *i.e., quantum proofs cannot substitute for interaction.*

Main result

$$
\begin{tikzcd}
\mathcal{IPP} \arrow[rr, red, "/" marking] & & \mathcal{QMAP} \arrow[d, red, shift left, "/" marking] \\
& & \mathcal{QCMAP} \arrow[u, shift left] \\
\mathcal{MAP} \arrow[uu] \arrow[urr] \arrow[rr, red, shift left, "/" marking] & & \mathcal{QPT} \arrow[u] \arrow[ll, red, shift left, "/" marking] \\
& \mathcal{PT} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$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Techniques

$\mathcal{MAP} \not\subseteq \mathcal{QPT}$: disjointness + relaxed locally decodable code

$\mathcal{QPT} \not\subseteq \mathcal{MAP}$: Forrelation [Aaronson and Ambainis, 2018]

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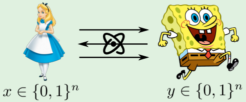
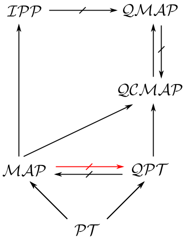


Given $C(x)$ and $C(y)$, $\exists i \in [n]$ such that $x_i = y_i = 1$?



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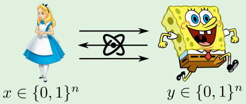
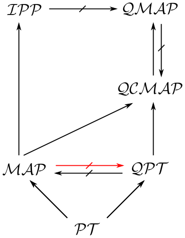


Given $C(x)$ and $C(y)$, $\exists i \in [n]$ such that $x_i = y_i = 1$?

- $\Omega(\sqrt{n})$ without proof
- $O(1)$ with $\log n$ proof



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Given $f, g : \{0,1\}^{\log n} \to \{0,1\}$, is $\langle f, \hat{g} \rangle$ small?

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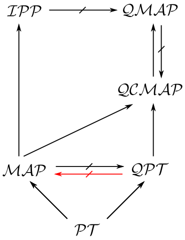

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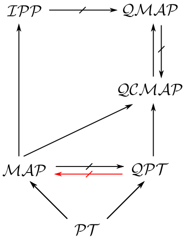
- $O(1)$ quantum, without proof
- $p \cdot q = \Omega(n^{1/4})$ classical, with proof

$\mathcal{QMAP} \not\subseteq \mathcal{QCMAP}$: recasting $\mathcal{QMA} \not\subseteq \mathcal{QCMA}$ [Aaronson and Kuperberg, 2007]

$\mathcal{IPP} \not\subseteq \mathcal{QMAP}$: permutation testing [Gur et al., 2018, Sherstov and Thaler, 2019]



Techniques

$\mathcal{MAP} \not\subseteq \mathcal{QPT}$: disjointness + relaxed locally decodable code

$\mathcal{QPT} \not\subseteq \mathcal{MAP}$: Forrelation [Aaronson and Ambainis, 2018]

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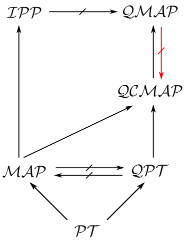

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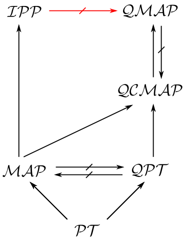

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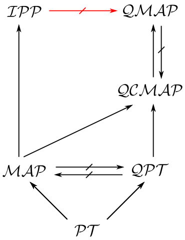

Given $f : [n] \to [n]$, is f a permutation?

- $O(1)$ with (classical) interaction
- $p \cdot q = \Omega(n^{1/3})$ quantum, without interaction

Open problems

- What about $\mathcal{QIPP}$?

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Thank you!

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