A structural theorem for local algorithms with applications to coding, testing and privacy

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An efficient transformation from local to sample-based algorithms.

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Local: inspects a small part of the input string x
Sample-based: access by random (i, x_i) samples
[Goldreich and Ron, 2016]

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But why?

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- Local codes
- PCPs and PCPPs
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Local + robust
($O(1)$ queries)

Local + sample-based
($o(n)$ queries)

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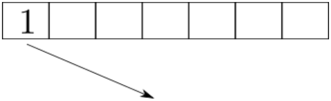

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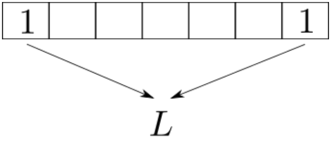

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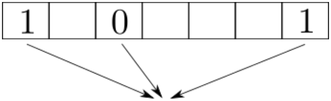

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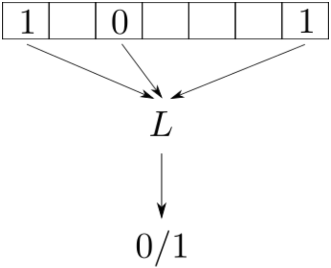

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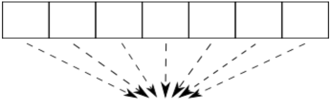

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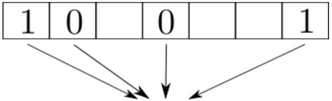

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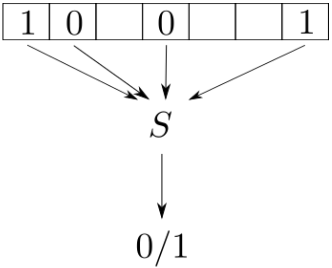
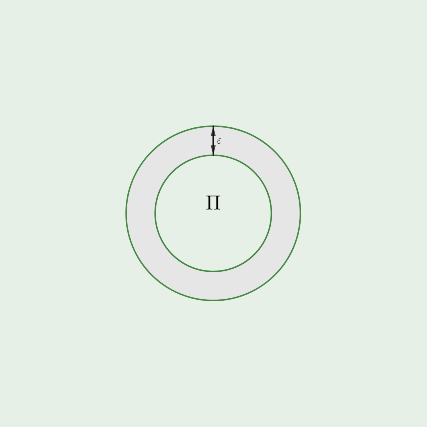


Property tester

Property $\Pi \subseteq \{0,1\}^n$, proximity parameter $\varepsilon > 0$



Tester T computes

$$f(x) = \begin{cases} 1, & \text{if } x \in \Pi \\ 0, & \text{if } x \text{ is } \varepsilon\text{-far from } \Pi. \end{cases}$$

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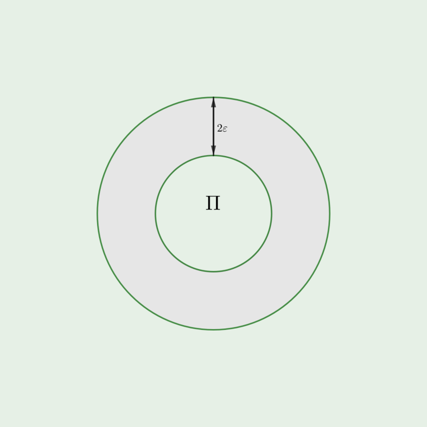

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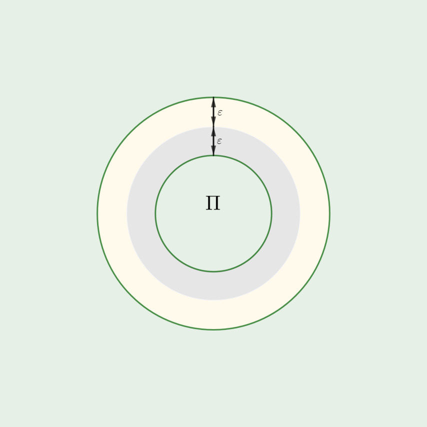

Tester T **robustly** computes

$$g(x) = \begin{cases} 1, & \text{if } x \in \Pi \\ 0, & \text{if } x \text{ is } 2\varepsilon\text{-far from } \Pi. \end{cases}$$

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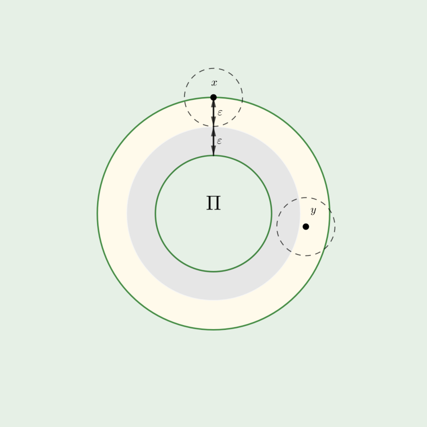

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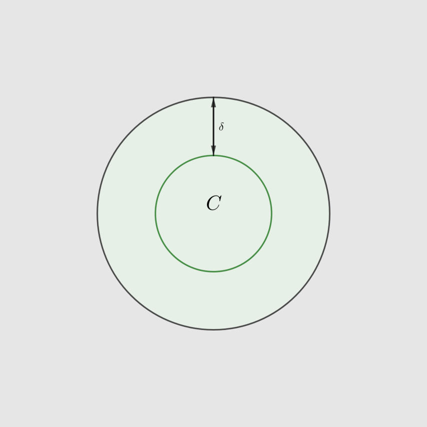
Code $C : \{0,1\}^k \to \{0,1\}^n$, decoding radius $\delta > 0$



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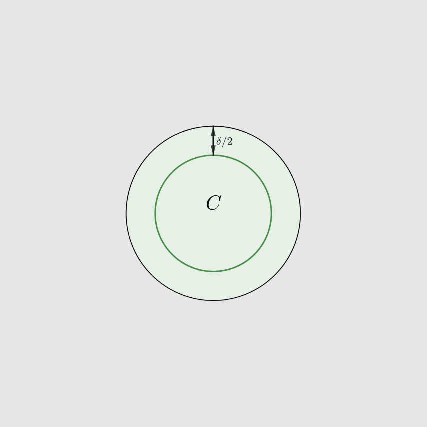

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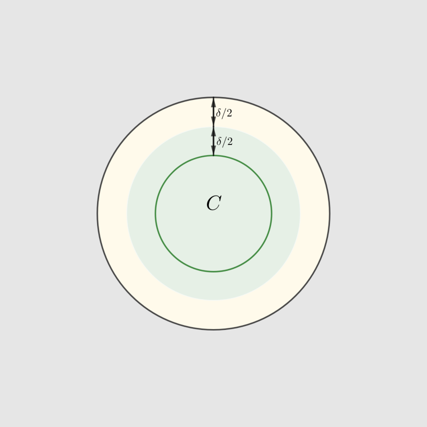

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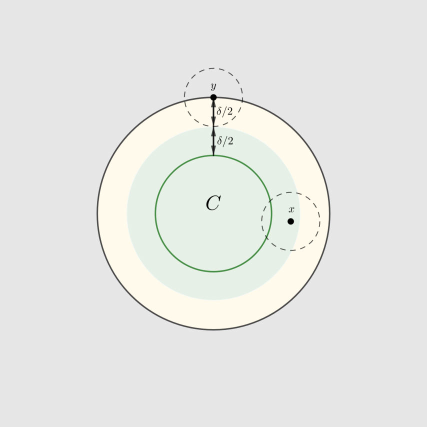

Decoder D **robustly** computes

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Definition

Let $X \subset \{0,1\}^n$. An algorithm L computes $f : X \times Z \to \{0,1\}$ with error rate σ if

$$\mathbb{P}[L^x(z) = f(x,z)] \geq 1 - \sigma. \qquad (\forall x \in X)$$

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Remarks:

- Robust 0-inputs without loss of generality.
- f is either partial or constant.
- Captures **LTCs, LCCs, MAPs, PCPPs**...

Theorem

Any function computed by an $\Omega(1)$*-robust local algorithm with query complexity* q *admits a sample-based algorithm with sample complexity*

$$n^{1-\Omega(1/(q^2 \log^2 q))}.$$

$$(q = \Omega(\sqrt{\log n}) \implies \text{sample complexity } \Omega(n))$$

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Theorem

This transformation cannot achieve sample complexity

$$n^{1-\omega(1/q)}.$$

Applications ($q = O(1)$)

Improved lower bound on the blocklength of a *relaxed* LDC.
State-of-the-art was $n = k^{1+\tilde{\Omega}\left(1/2^{2q}\right)}$ [Gur and Lachish, 2020].

Corollary

Any code $C : \{0,1\}^k \to \{0,1\}^n$ that is relaxed locally decodable with q queries satisfies

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([Asadi and Shinkar, 2020] achieves $n = k^{1+O(1/q)}$, improving on the construction of [Ben-Sasson et al., 2004])

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Known "P vs. NP separation for testing" is essentially optimal.
$\exists$ property testable with q queries and $O(\log n)$-long proof, while $n^{1-O(1/q)}$ are needed without a proof [Gur and Rothblum, 2018].

Corollary

If a property is ε-testable with a short proof and q queries, then it is 2ε-testable with $n^{1-\tilde{\Omega}(1/q^2)}$ queries and no proof. In particular, a $O(1)$ vs. $\Omega(n)$ separation is impossible.

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(short: sublinear in the sample complexity)

Applications ($q = O(1)$)

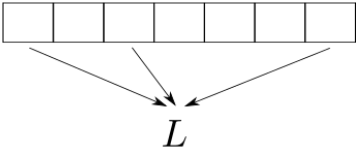
Extension of [Fischer et al., 2015] to adaptive testers.

Corollary

Any property ε-testable with q queries admits a sample-based 2ε-tester with sample complexity $n^{1-\tilde{\Omega}(1/q^2)}$.

Warm-up: a special case

L **non-adaptive**, one-sided and its query sets form a sunflower



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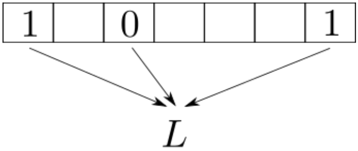

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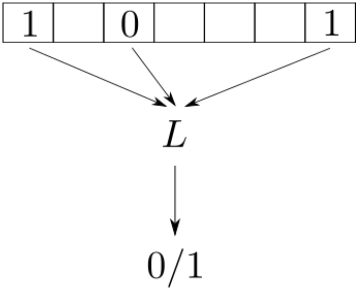

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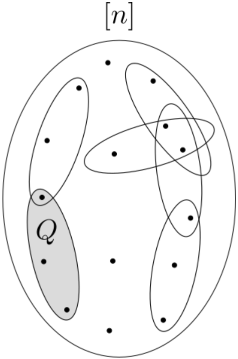


$Q \in \mathcal{Q}$ sampled with probability $\mu(Q)$

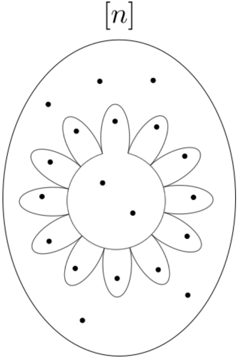
Warm-up: a special case

L non-adaptive, **one-sided** and its query sets form a sunflower

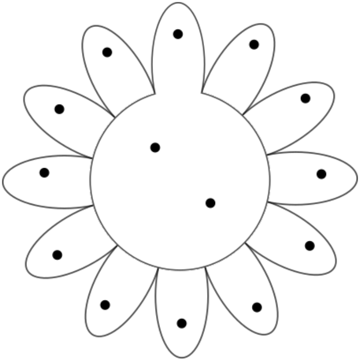
If $f(x) = 1$, then L^x outputs 1 *with certainty*.

Warm-up: a special case

L non-adaptive, one-sided and its **query sets form a sunflower**



Goal: sample many query sets of L and aggregate its "votes"
$(p \approx 1/n^{\alpha})$



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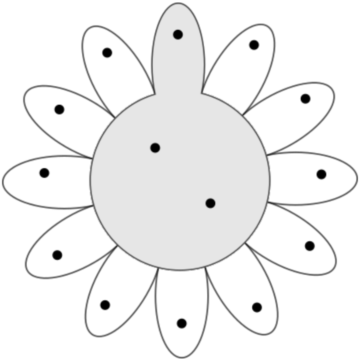

Query at least one set with probability $\leq p$ ☹

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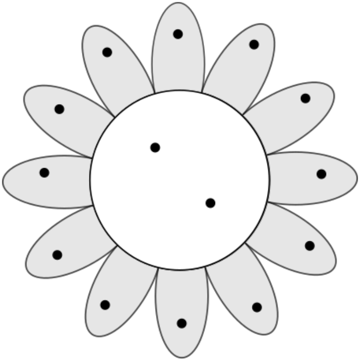
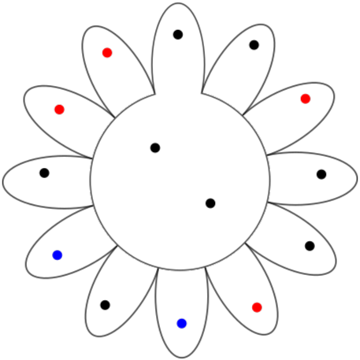
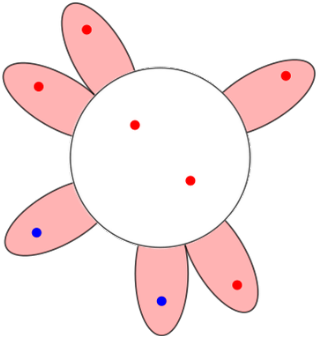


Query $\approx p|\mathcal{Q}|$ *partial* sets with high probability ☺
Fill in the kernel arbitrarily – robustness!

Output 1 $\iff$ some kernel assignment yields *all votes* for 1



($\bullet = 0$, $\bullet = 1$; $\circ$ = vote for 0, $\circ$ = vote for 1)



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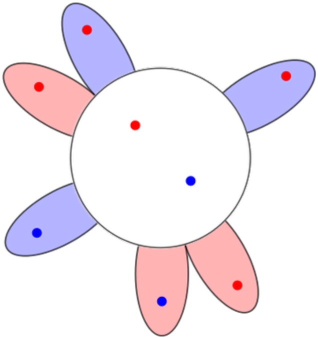

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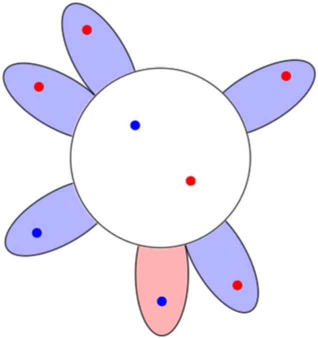

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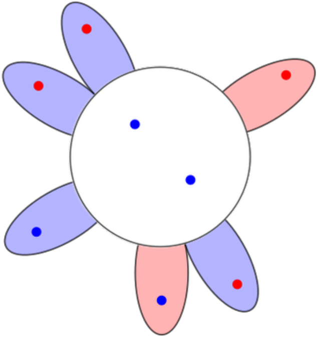

Output 0!

There isn't always a sunflower...

But there is always a daisy!

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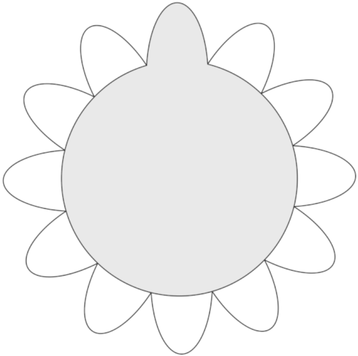

$|\mathcal{S}| = o(|\mathcal{Q}|)$

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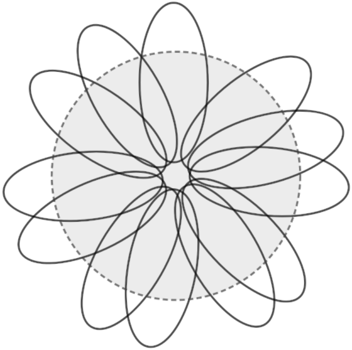

$|\mathcal{D}| = \Omega(|\mathcal{Q}|)$

The bottlenecks

With error $\sigma \approx 1/q$, some daisy approximates L and we throw away the rest. [Gur and Lachish, 2020]

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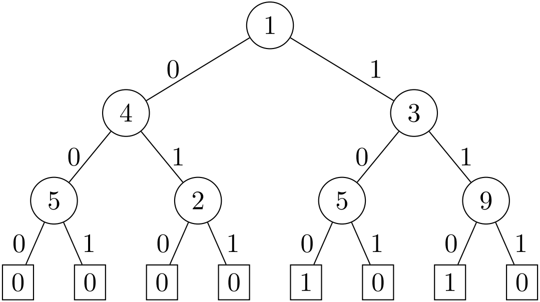
Adaptivity: decision trees vs. query sets. Some daisy works, but we **don't know which**!

Two-sided error: no hope for a consensus.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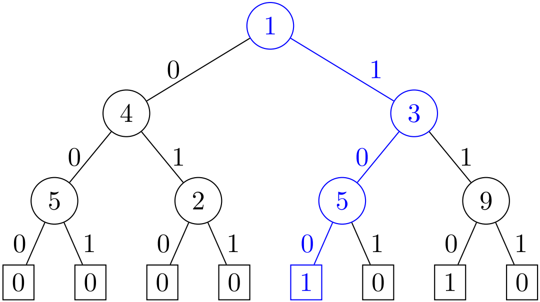
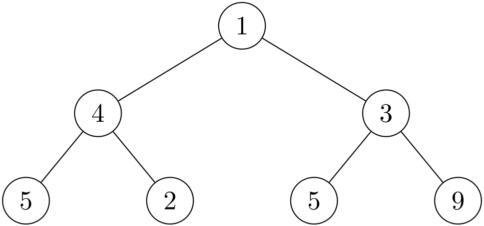
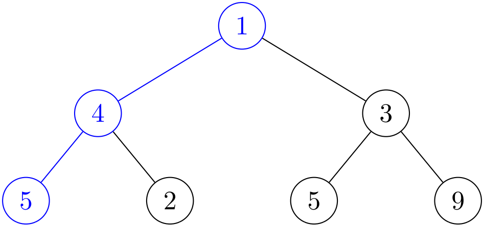


$$x_1 = 1$$
$$x_3 = 0$$
$$x_5 = 0$$



$$\mathcal{Q}_T = \left\{ \quad \left\{ \begin{array}{c} 1 \\ 4 \\ 5 \end{array} \right\}, \quad \left\{ \begin{array}{c} 1 \\ 4 \\ 2 \end{array} \right\}, \quad \left\{ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} \right\}, \quad \left\{ \begin{array}{c} 1 \\ 3 \\ 9 \end{array} \right\} \quad \right\}$$



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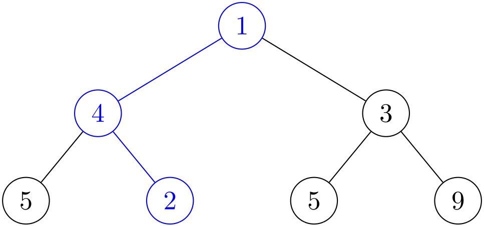

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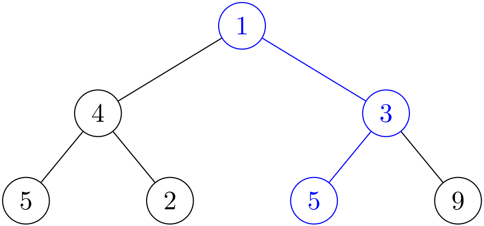

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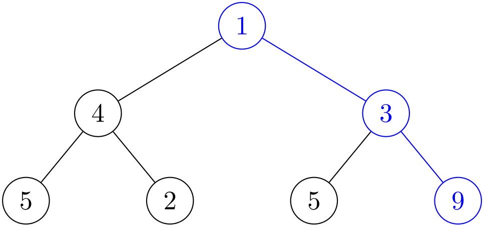

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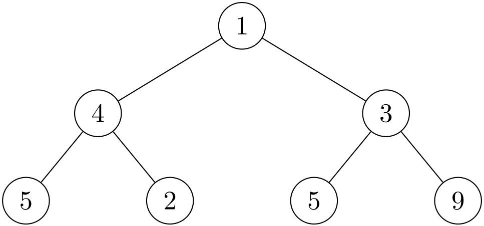
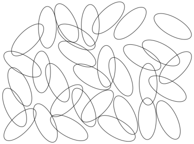


$$\mathcal{Q}_T = \left\{ \left\{ \begin{array}{c} 1 \\ 4 \\ 5 \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ 4 \\ 2 \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} \right\}, \left\{ \begin{array}{c} 1 \\ 3 \\ 9 \end{array} \right\} \right\}$$

$$\mathcal{Q} = \bigcup_T \mathcal{Q}_T$$

Daisy partition theorem

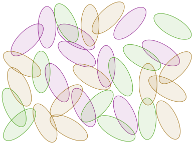


$\mathcal{Q}$

An *arbitrary* $\mathcal{Q}$ ($|\mathcal{Q}| \approx n$) can be partitioned into $\mathcal{D}_0, \ldots, \mathcal{D}_q$.
$\mathcal{D}_i$ has:

- petals of size i
- small kernel $|K_i|$
- small intersection

Daisy partition theorem



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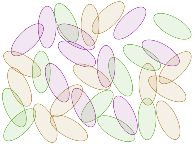
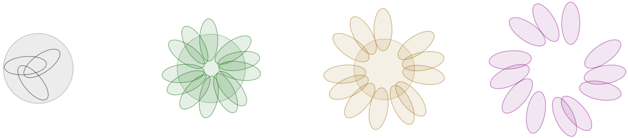


$\downarrow$



$\mathcal{D}_0$ $\mathcal{D}_1$ $\mathcal{D}_2$ $\mathcal{D}_3$

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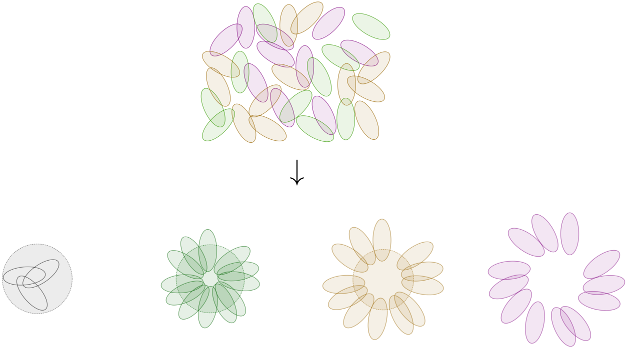

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$\mathcal{D}_i$ has:

- petals of size i
- small kernel $|K_i|$ ($\implies$ few kernel assignments)
- small intersection (+ Hajnal-Szemerédi $\implies$ many sampled petals)

The algorithm

1. Sample each element of $[n]$ with probability p.
2. For every $i \in [q]$ and assignment κ_i to K_i:
   If L votes 1 on $\geq \tau_i$ sets with sampled petals, output 1.
3. Otherwise, output 0.

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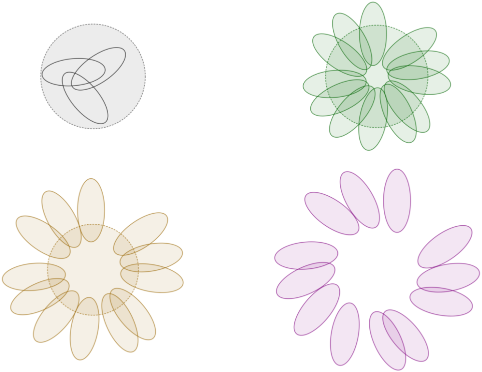

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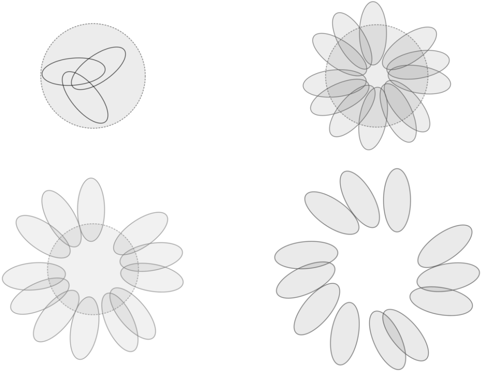

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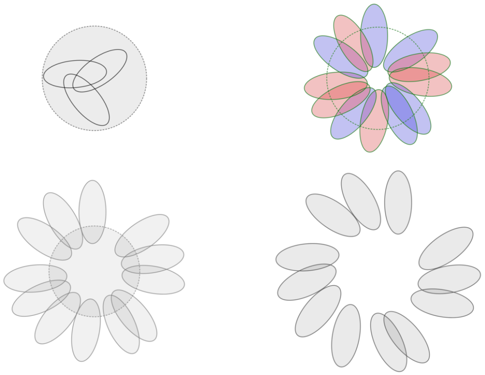

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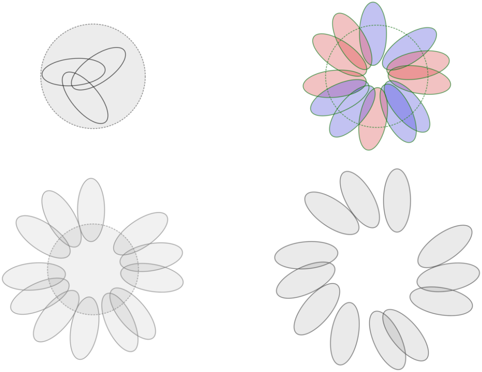

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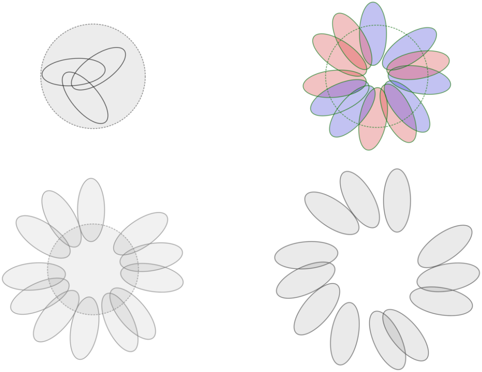

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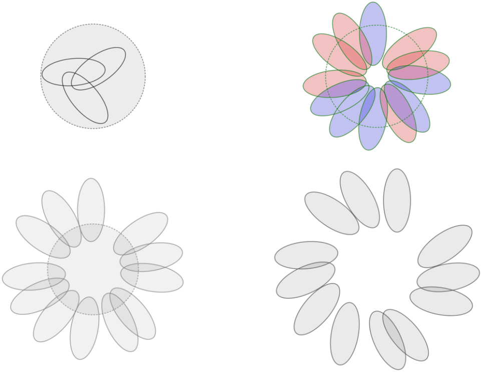

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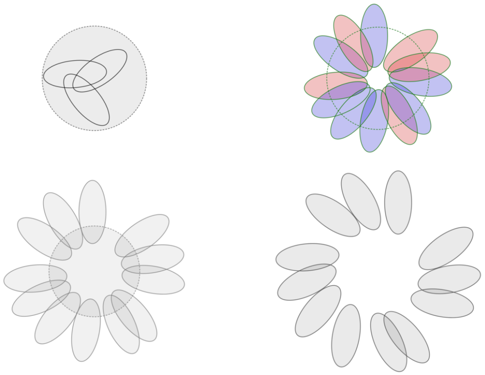

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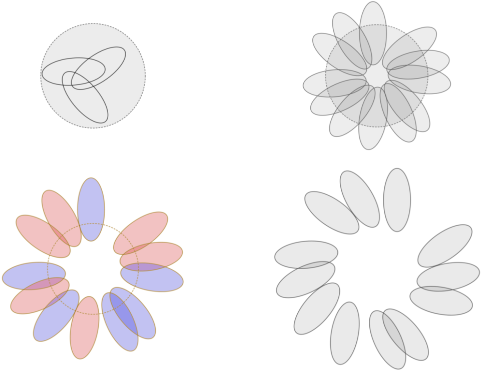

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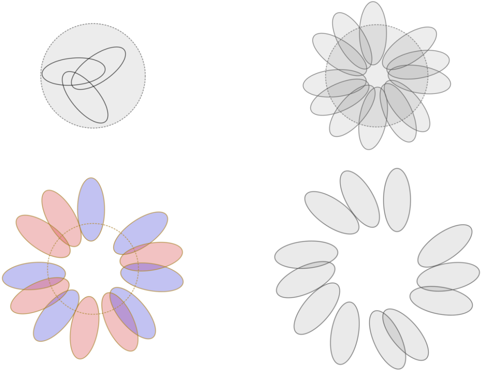

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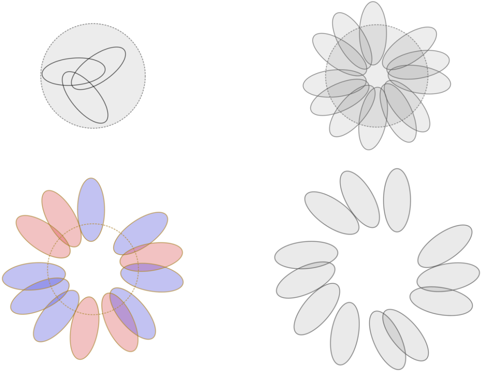

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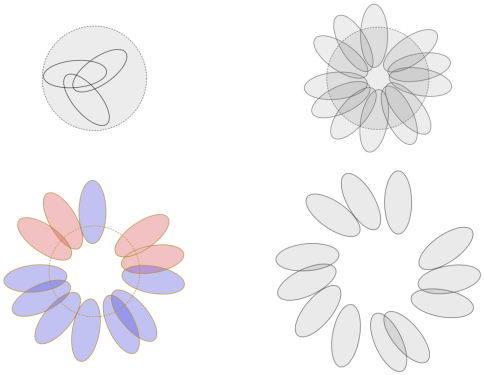

Open problems

- Closing the quadratic gap between RLDC blocklength lower and upper bounds ($k^{1+\tilde{\Omega}(1/q^2)}$ vs. $k^{1+O(1/q)}$).
- PCPP lower bounds by similar techniques?
- Capturing, e.g., PAC learning or local computation algorithms?

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