A structural theorem for local algorithms with applications to coding, testing and privacy

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HALG 2022

Main result

Techniques

Applications

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Applications

An efficient transformation from local to sample-based algorithms.

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Local: inspects a small part of the input string x
Sample-based: access by random (i, x_i) samples

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But why?

Sample-based access buys

- Privacy

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- Efficient repetition: running q-query $L_1, \ldots, L_t$ on the same input takes
  - $O(qt \log t)$ queries in general
  - $O(q \log t)$ queries by **reusing samples**

Local + robust ($O(1)$ queries)		Local + sample-based ($o(n)$ queries)
L	$\Longrightarrow$	S

Local $+$ robust
($O(1)$ queries)

Local $+$ sample-based
($o(n)$ queries)

$$L \quad \Longrightarrow \quad S$$

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Local + sample-based
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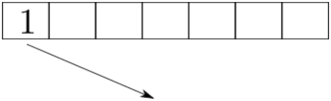

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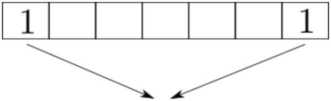

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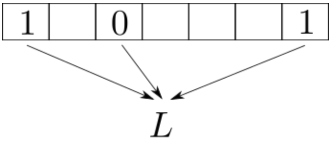

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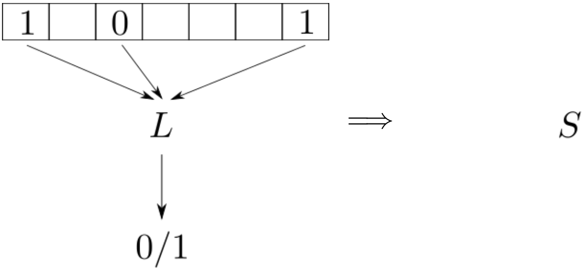

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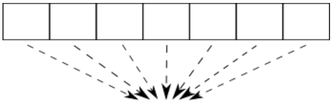

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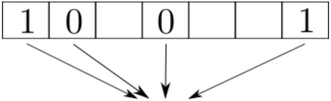

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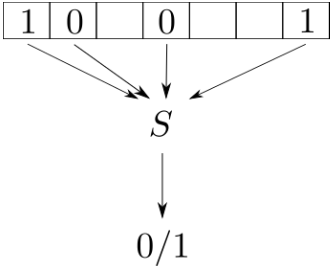
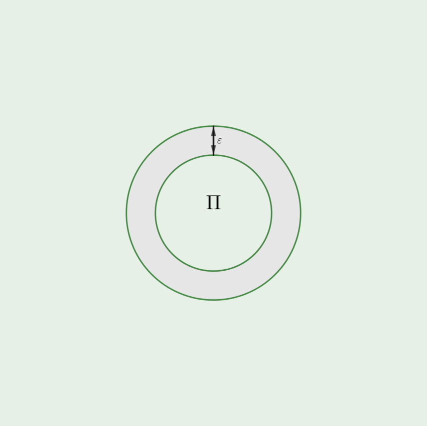


Property tester

Property $\Pi \subseteq \{0,1\}^n$, proximity parameter $\varepsilon > 0$



Tester T computes

$$f(x) = \begin{cases} 1, & \text{if } x \in \Pi \\ 0, & \text{if } x \text{ is } \varepsilon\text{-far from } \Pi. \end{cases}$$

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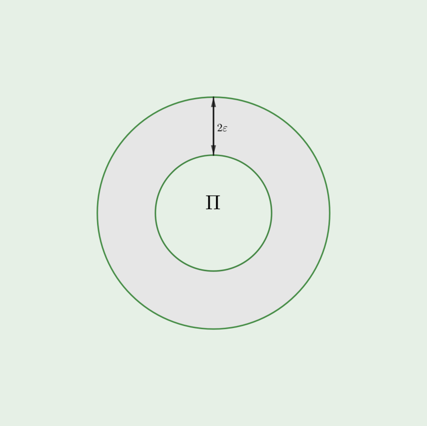

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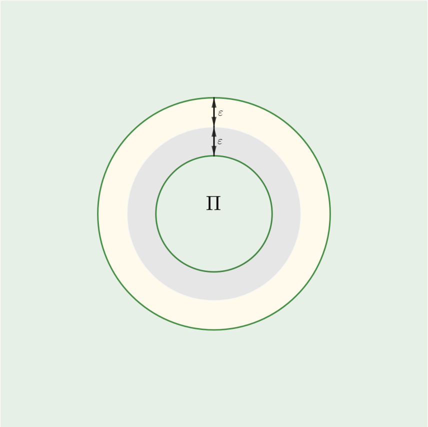

Tester T **robustly** computes

$$g(x) = \begin{cases} 1, & \text{if } x \in \Pi \\ 0, & \text{if } x \text{ is } 2\varepsilon\text{-far from } \Pi. \end{cases}$$

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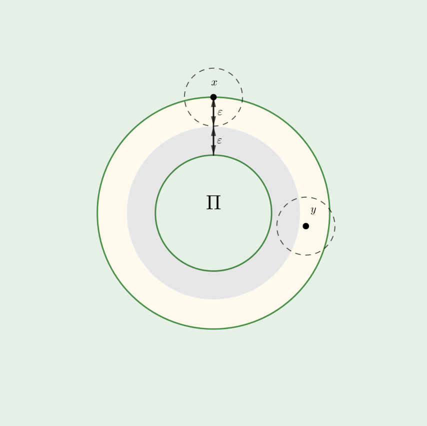

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Theorem

Any function computed by a q-query robust local algorithm admits a sample-based algorithm with sample complexity

$$n^{1-\tilde{\Omega}(1/q^2)}.$$

$$(q = \Omega(\sqrt{\log n}) \implies \text{sample complexity } \Omega(n))$$

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Theorem

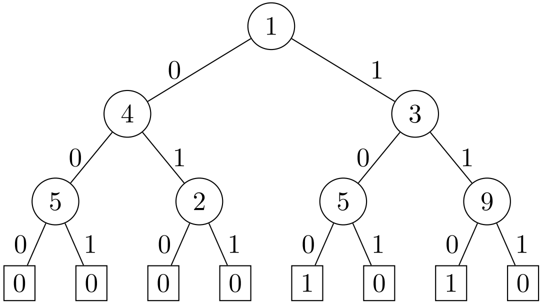
This transformation cannot achieve sample complexity

$$n^{1-\omega(1/q)}.$$

Main result

Techniques

Applications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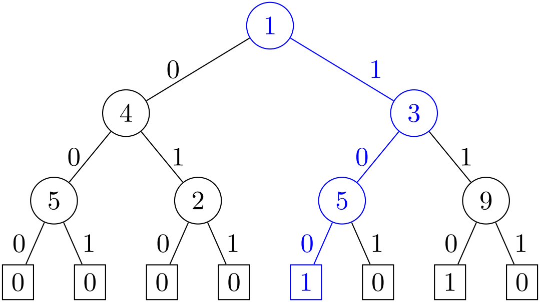
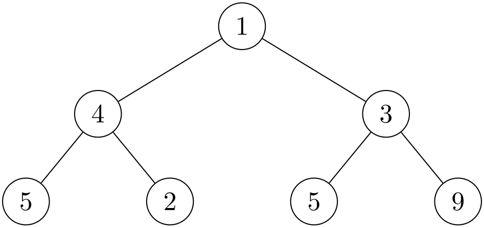
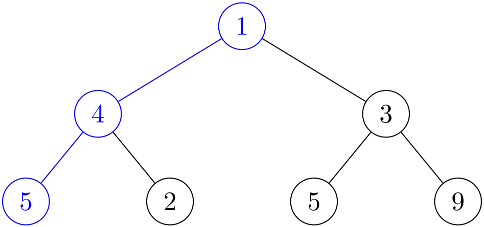


$$x_1 = 1$$
$$x_3 = 0$$
$$x_5 = 0$$



$$\mathcal{Q}_T = \left\{ \left\{ \begin{matrix} 1 \\ 4 \\ 5 \end{matrix} \right\}, \left\{ \begin{matrix} 1 \\ 4 \\ 2 \end{matrix} \right\}, \left\{ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} \right\}, \left\{ \begin{matrix} 1 \\ 3 \\ 9 \end{matrix} \right\} \right\}$$



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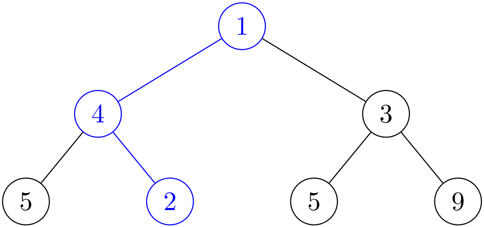

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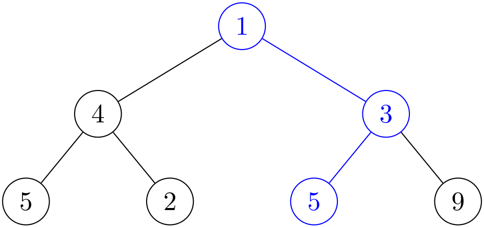

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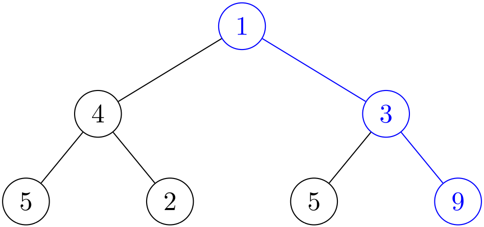

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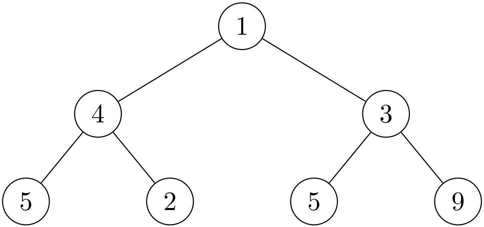

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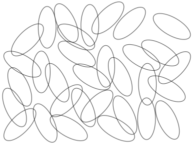
$$\mathcal{Q} = \bigcup_T \mathcal{Q}_T$$

Daisy partition theorem



$\mathcal{Q}$

An *arbitrary* $\mathcal{Q}$ ($|\mathcal{Q}| \approx n$) can be partitioned into $\mathcal{D}_0, \ldots, \mathcal{D}_q$.
$\mathcal{D}_i$ has:

- petals of size i
- small kernel $|K_i|$
- bounded petal intersection

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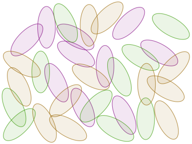

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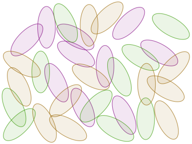
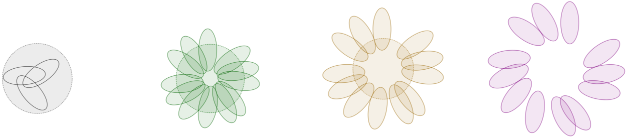


$\downarrow$



$\mathcal{D}_0$ $\mathcal{D}_1$ $\mathcal{D}_2$ $\mathcal{D}_3$

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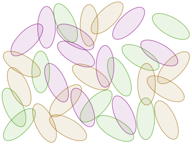

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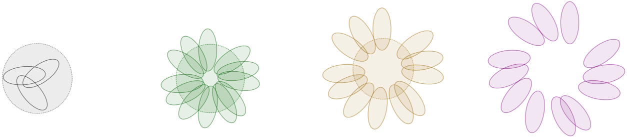

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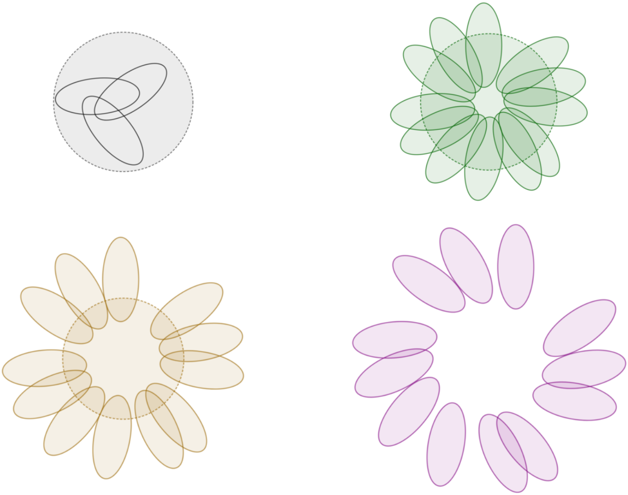
Adaptive-to-sample-based transformation for testers

q-query tester $\implies$ $n^{1-\tilde{\Omega}(1/q^2)}$-query sample-based tester.



Thank you!